Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
() : –

doi:10.55730/1300-0098.

Dynamics of compositions of some Lotka–Volterra mappings operating in a two-dimensional simplex

Dilfuza ESHMAMATOVA*, **Farrukh YUSUPOV**
Department, Tashkent State Transport University,Tashkent, Uzbekistan

Received: 27.02.2023 • **Accepted/Published Online:** 17.02.2024 • **Final Version:** ..202

Abstract: In a number of applied problems (immunology, epidemiology, virology, etc.), there is a need to study the dynamics of the trajectories of the compositions of the Lotka–Volterra mappings. In this paper, some variants of these compositions are considered, in particular, the dynamics of the trajectories of the compositions of the Lotka–Volterra mappings acting in a two-dimensional simplex, with transitive tournaments, which can be applied in the study of these processes. Cards of fixed points are constructed for compositions and the characters of fixed points are studied.

Key words: Lotka–Volterra mapping, transitive tournament, fixed point, repeller, attractor

1. Introduction

The task of the qualitative theory of dynamical systems is to develop methods that allow us to study the behavior of the trajectories of the system in the entire domain of its task (for discrete dynamical systems without system integration). The main step of these studies is to study the behavior of the trajectories of the system in the vicinity of each of its singular points. It is known that the founders of the qualitative theory of differential equations, i.e. continuous dynamical systems, are the famous French mathematician Jules Henri Poincare (1854–1912) and the famous Russian mathematician Alexander Mikhailovich Lyapunov (1857–1918). These scientists are responsible for the formulation of initial tasks, fruitful ideas for their solutions, and fundamental concrete results that have received wide resonance in the scientific world. Their first followers were I. Bendikson (1861–1920), A. Dulak (1870–1955), O. Perron (1880–1975), D. Birkhoff (1884–1944), as well as V. V. Stepanov (1889–1950), I. G. Petrovsky (1901–1973), N. G. Chetaev (1902-1959), and others [1].

It is known that until the beginning of the 20th century, the field of natural science that fed the qualitative theory of differential equations was celestial mechanics [1], but by the beginning of the 20th century, the situation had changed significantly. The theory of dynamical systems began to be applied in various fields of physics, mechanics, optics, acoustics, as well as in population genetics, epidemiology, and environmental problems. Despite numerous works in the theory of dynamical systems, quite a lot of questions remain open in this area. In this paper, we consider the dynamics of the trajectories of compositions of quadratic stochastic Lotka–Volterra maps acting in a two-dimensional simplex.

Let $V : \mathbb{R}^m \to \mathbb{R}^m$ be a quadratic stochastic operator, defined by specifying a cubic matrix $\{P_{ij,k}\}_{i,j,k=\overline{1,m}}$, whose coefficients satisfy the conditions [4, 5].

*Correspondence: 24dil@mail.ru
2010 *AMS Mathematics Subject Classification:* 37B25, 37C25, 37C27.

$$P_{ij,k} = P_{ji,k} \geq 0, \sum_{k=1}^{m} P_{ij,k} = 1$$

and it works according to the formulas:

$$Vx = \left(\sum_{i,j=1}^{m} P_{ij,1} x_i x_j, \sum_{i,j=1}^{m} P_{ij,2} x_i x_j, ..., \sum_{i,j=1}^{m} P_{ij,m} x_i x_j \right), \tag{1.1}$$

where $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$.

It is known that the quadratic stochastic operator leaves the hyperplane invariant $H = \left\{ x \in \mathbb{R}^m : \sum\limits_{i=1}^{m} x_i = 1 \right\}$, as well as the basic simplex $S^{m-1} = \left\{ x \in \mathbb{R}^m : x_i \geq 0, \sum\limits_{i=1}^{m} x_i = 1 \right\}$.

Definition 1.1 *[4] A quadratic stochastic operator is called a Lotka–Volterra mapping if* $P_{ij,k} = 0$ *by* $k \notin \{i, j\}$.

Then [4] the Lotka–Volterra mapping, acting in the simplex S^{m-1}, can always be represented as

$$x'_k = x_k(1 + \sum_{i=1}^{m} a_{ki} x_i), k = \overline{1, m}, \tag{1.2}$$

$a_{ki} = -a_{ik}, |a_{ki}| \leq 1$.

Theorem 1.2 *[6] The mapping* $V : S^{m-1} \to S^{m-1}$*, defined by formula (1.2), is a homeomorphism, and for* $| a_{ki} | < 1$ *for all* $k, i = \overline{1, m}$ $-$ *by a diffeomorphism of the simplex* S^{m-1}.

For an arbitrary starting point $x^0 \in S^{m-1}$ the sequence $\{x^{(n)}\} \subset S^{m-1}$, defined by the recurrent formula

$$x^{(n+1)} = Vx^{(n)}, \quad n = 0, 1, ...,$$

it is called a trajectory starting from a point x^0.

Through $\omega(x^0) = \{x^0, x^{(1)}, ...\}'$, we denote the set of limit points of a positive trajectory. Obviously, $\omega(x^0)$ $-$ is a nonempty closed and invariant subset S^{m-1}, i.e. $V(\omega(x^0)) \subset \omega(x^0)$.

Since V $-$ is a homeomorphism for $|a_{ki}| \leq 1$, there is a negative trajectory for any inner point $x^0 \in S^{m-1}$ of the simplex

$$x^{(-n-1)} = V^{-1}(x^{(-n)}), \ n = 0, 1, ...$$

Through $\alpha(x^0) = \{x^0, x^{(-1)}, ..., x^{(-n)}, ...,\}'$, denote the set of limit points of the negative trajectory.

For the further presentation of the work, we will need several definitions from graph theory [9, 10, 12].

Definition 1.3 *[9] The graph* G $-$ *is a finite nonempty set* W *containing* p *vertices and a given set* E *containing* q *disordered pairs of distinct vertices from* W.

Each pair of $x = \{u, v\}$ vertices in E will be called an edge of the graph G, and this will mean that x connects u and v. We will write $x = uv$, this will mean that u and v – adjacent vertices. The vertex u and the edge x are incident, as are v and x.

If two different edges x and y are incident to the same vertex, then they are called adjacent.

A graph with p vertices and q edges is called a (p, q)-graph.

It is clear from the definitions that there can be no loops in the graph, that is, edges connecting the vertices with themselves.

Definition 1.4 *[9] A directed graph or digraph D – is a finite nonempty set of vertices and a given set of ordered pairs of distinct vertices.*

The elements of the set E are called oriented edges or arcs.

Definition 1.5 *Pairs of vertices that are connected by more than one edge are called multiples.*

There are no loops and multiple arcs in the digraph.

Definition 1.6 *A directed graph is a digraph in which no pair of vertices is connected by a symmetric pair of arcs.*

It follows from the definition that every orientation of a graph generates a directed graph.

Each mapping we consider corresponds to a tournament; therefore, along with system (1.2), we consider a complete graph called a tournament and introduce it as follows:

Let $A = (a_{ki})$ – skew-symmetric matrix, i.e. $A' = -A$, where A' – the matrix transposed to A [7].

Assuming that $a_{ki} \neq 0$ for $k \neq i$, we construct a tournament T_m with vertices $1, 2, ..., m$ as follows: if $a_{ki} > 0$, then we connect the vertices k and i an arrow (edge) directed from i–that vertex to k–that vertex. Next, the constructed tournament T_m will be called the tournament of the dynamical system (1.2) with the skew-symmetric matrix $A = (a_{ki})$.

The transitivity of the tournament means that any subtournament of this tournament is not strong [2, 3, 8].

The purpose of this work is to study the dynamics of the composition of Lotka–Volterra mappings acting in S^2 with transitive tournaments, with one mutually-inversely directed edge. It is known [5, 6] that when the Lotka–Volterra mapping is in general position, we can introduce the concept of a tournament, and to study the dynamics of the behavior of the trajectories of the composition, we will introduce the concept of a map of fixed points. To do this, we need the following Lemma.

Lemma 1.7 *[6] Let $A = (a_{ki})$ – skew-symmetric matrix. Then the solution of the system of linear inequalities*

$$P = \{x \in S^{m-1} : \sum_{i=1}^{m} a_{ki} x_i \geq 0, k = \overline{1, m}\}$$

and

$$Q = \{x \in S^{m-1} : \sum_{i=1}^{m} a_{ki} x_i \leq 0, k = \overline{1, m}\}$$

– convex nonempty polyhedra.

2. Main results

Let two Lotka–Volterra mappings acting in a two-dimensional simplex S^2 :

$$V_1: \begin{cases} x_1' = x_1(1 + a_{12}x_2 + a_{13}x_3), \\ x_2' = x_2(1 - a_{12}x_1 - a_{23}x_3), \\ x_3' = x_3(1 - a_{13}x_1 + a_{23}x_2), \end{cases} \quad V_2: \begin{cases} x_1' = x_1(1 + b_{12}x_2 + b_{13}x_3), \\ x_2' = x_2(1 - b_{12}x_1 + b_{23}x_3), \\ x_3' = x_3(1 - b_{13}x_1 - b_{23}x_2), \end{cases} \tag{2.1}$$

here $|a_{ki}| \leq 1, a_{ki} = -a_{ik}, |b_{ki}| \leq 1, b_{ki} = -b_{ik}, \sum_{i=1}^{3} x_i = 1$.

Consider the composition of these mappings. Each of these mappings is an automorphism of the simplex, and it is also obvious from this that the composition $V_1 \circ V_2$ is an automorphism of the simplex S^2, and it is representable as:

$$V_1 \circ V_2: \begin{cases} x_1' = x_1(1 + f_1(x_2, x_3)), \\ x_2' = x_2(1 + f_2(x_1, x_3)), \\ x_3' = x_3(1 + f_3(x_1, x_2)), \end{cases} \tag{2.2}$$

where the functions f_1, f_2, f_3 are polynomials of the third degree of x_1, x_2, x_3, satisfying the condition

$$x_1 \cdot f_1 + x_2 \cdot f_2 + x_3 \cdot f_3 \equiv 0.$$

If we describe the system (2.2) in detail, we will get the following picture:

$$V_1 \circ V_2: \begin{cases} x_1' = x_1(1 + b_{12}x_2 + b_{13}x_3)(1 + a_{12}x_2(1 - b_{12}x_1 + b_{23}x_3) + a_{13}x_3(1 - b_{13}x_1 - b_{23}x_2)), \\ x_2' = x_2(1 - b_{12}x_1 + b_{23}x_3)(1 - a_{12}x_1(1 + b_{12}x_2 + b_{13}x_3) - a_{23}x_3(1 - b_{13}x_1 - b_{23}x_2)), \\ x_3' = x_3(1 - b_{13}x_1 - b_{23}x_2)(1 - a_{13}x_1(1 + b_{12}x_2 + b_{13}x_3) + a_{23}x_2(1 - b_{12}x_1 + b_{23}x_3)). \end{cases} \tag{2.3}$$

$$V_2 \circ V_1: \begin{cases} x_1' = x_1(1 + a_{12}x_2 + a_{13}x_3)(1 + b_{12}x_2(1 - a_{12}x_1 - a_{23}x_3) + b_{13}x_3(1 - a_{13}x_1 + a_{23}x_2)), \\ x_2' = x_2(1 - a_{12}x_1 - a_{23}x_3)(1 - b_{12}x_1(1 + a_{12}x_2 + a_{13}x_3) + b_{23}x_3(1 - a_{13}x_1 + a_{23}x_2)), \\ x_3' = x_3(1 - a_{13}x_1 + a_{23}x_2)(1 - b_{13}x_1(1 + a_{12}x_2 + a_{13}x_3) - b_{23}x_2(1 - a_{12}x_1 - a_{23}x_3)). \end{cases} \tag{2.4}$$

Let $I = \{1, ..., m\}$ and $\alpha \subset I$, $X = \{x(\alpha) : \alpha \subset I\}-$ a set of fixed points V. Since $V : S^{m-1} \to S^{m-1}$ is continuous, and S^{m-1} is a convex compact, then according to the Bohl-Brauer theorem, the set of fixed points V is nonempty. We will say that the fixed points are $x(\alpha)$ and $x(\beta)$ form a pair (p, q), if there exists a face Γ_γ such that $\gamma \supset \alpha \cup \beta$, and the are satisfied inequalities $A_\gamma x(\alpha) \geq 0, A_\gamma x(\beta) \leq 0$ (according to the Lemma 1.7). In this case, $x(\alpha)$ let us call p a point, and $x(\beta)$ – q a point. Now we will represent the elements of X as points on the plane and if $x(\alpha)$ and $x(\beta)$ form the pair of (p, q), then we connect them with an arc, that is, an arrow directed from $x(\alpha)$ to $x(\beta)$. The resulting oriented graph is called the card of fixed points of the V mapping.

Lemma 2.1 *The card of fixed points of the composition of the Lotka–Volterra mappings $V_1 \circ V_2$ and $V_2 \circ V_1$, acting in S^2, is isomorphic to one of the following three cards, shown in Figure 1.*

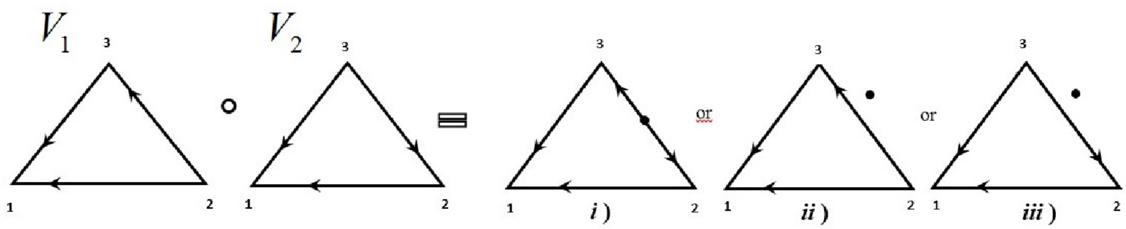

Figure 1. Figure 1. Card of fixed points of the composition $V_1 \circ V_2$ and $V_2 \circ V_1$, acting in S^2.

To prove the Lemma, we will consider several special cases of compositions of Lotka–Volterra mappings defined by the equalities (2.4). The generalized case requires rather capacious calculations, as it is apparent from expressions (2.4) that six parameters are obtained with arbitrary coefficients in the system. For each case, we have obtained theorems and statements with which we will prove Lemma 2.1. Since the picture of the phase portrait turns out to be interesting, let us consider the dynamics of the composition first with the introduction of one coefficient. For example,

$$V_3: \begin{cases} x' = x(1 + ay + z), \\ y' = y(1 - ax - z), \\ z' = z(1 - x + y), \end{cases} \quad V_4: \begin{cases} x' = x(1 + y + z), \\ y' = y(1 - x + z), \\ z' = z(1 - x - y), \end{cases} \tag{2.5}$$

The composition of these mappings looks like as follows:

$$V_3 \circ V_4: \begin{cases} x' = x(1 + y + z)(1 + ay(1 - x + z) + z(1 - x - y)), \\ y' = y(1 - x + z)(1 - ax(1 + y + z) - z(1 - x - y)), \\ z' = z(1 - x - y)(1 - x(1 + y + z) + y(1 - x + z)), \end{cases} \tag{2.6}$$

and

$$V_4 \circ V_3: \begin{cases} x' = x(1 + ay + z)(1 + y(1 - ax - z) + z(1 - x + y)), \\ y' = y(1 - ax - z)(1 - x(1 + ay + z) + z(1 - x + y)), \\ z' = z(1 - x + y)(1 - x(1 + ay + z) - y(1 - ax - z)), \end{cases} \tag{2.7}$$

where the coefficient is $0 < a \leq 1$.

For qualitative study of the dynamics of the trajectory of the internal points of the composition $V_3 \circ V_4$ and $V_4 \circ V_3$, we find the fixed points of these compositions. First, we find a fixed point belonging to the edge Γ_{23}. To do this, in both cases, we take $x = 0$ and get the following systems of equations, respectively:

$$\begin{cases} 1 = (1 + z)(1 - z(1 - y)), \\ 1 = (1 - y)(1 + y(1 + z)), \\ 1 = y + z, \end{cases} , \begin{cases} 1 = (1 - z)(1 + z(1 + y)), \\ 1 = (1 + y)(1 - y(1 - z)), \\ 1 = y + z. \end{cases}$$

Solving the system data for composition (7), i.e. for $V_3 \circ V_4$, we get a fixed point

$$A_0\left(0;\frac{3-\sqrt{5}}{2};\frac{\sqrt{5}-1}{2}\right),$$

and for the composition $V_4 \circ V_3$, the fixed point has the following form:

$$A_1\left(0;\frac{\sqrt{5}-1}{2};\frac{3-\sqrt{5}}{2}\right).$$

Apart from to these fixed points for both compositions, all vertices of the simplex are stored as fixed points, i.e.

$$e_1(1;0;0), e_2(0;1,0), e_3(0;0;1).$$

In order to study the character of fixed points, we will use the Jacobi matrix and its spectrum. We will find the eigenvalues of the Jacobi matrix by solving the equation:

$$|J(x)-\lambda I|=0. \tag{2.8}$$

By the values of the eigenvalues, we can describe the character of fixed points. To do this, we first introduce definitions concerning the nature of fixed points [3].

Now, in order to investigate the character of the fixed points of the composition, we will introduce the following definitions:

Definition 2.2 *A fixed point is called an attracting point (attractor) if the spectrum of the Jacobian, i.e. the solution of equation (2.8), is modulo less than one.*

Definition 2.3 *A fixed point is called repulsive (repeller) if the spectrum of the Jacobian modulo is greater than one.*

Definition 2.4 *A fixed point is called a saddle point (i.e. it is neither a repeller nor an attractor) if among the solutions of equation (2.8) there are both modulo values greater than 1 and modulo values less than 1.*

Since we study hyperbolic systems, we do not consider the case when the eigenvalues are modulo 1.

The Jacobi matrix of operator (2.6) has the following form:

$$J(V_3\circ V_4)=\begin{pmatrix} A & B & C \\ D & K & L \\ M & N & O \end{pmatrix},$$

here

$$A = x(1+y+z)(-ay-z)+(1+y+z)(ay(1-x+z)+z(1-x-y)+1),$$

$$B = x(1+y+z)(a(1-x+z)-z)+x(ay(1-x+z)+z(1-x-y)+1),$$

$$C = x(1+y+z)(1-x-y+ay)+x(ay(1-x+z)+z(1-x-y)+1),$$

$$D = y(1-x+z)(z-a(1+y+z)) - y(-ax(1+y+z)+z(x+y-1)+1),$$

$$K = y(1-x+z)(z-ax) + (1-x+z)(-ax(1+y+z)+z(x+y-1)+1),$$

$$L = y(1-x+z)(-ax+x+y-1) + y(-ax(1+y+z)+z(x+y-1)+1),$$

$$M = z(1-x-y)(-2y-z-1) - z(y(1-x+z)-x(1+y+z)+1),$$

$$N = z(1-x-y)(1-2x+z) - z(y(1-x+z)-x(1+y+z)+1),$$

$$O = z(1-x-y)(y-x) + (1-x-y)(y(1-x+z)-x(1+y+z)+1).$$

The Jacobian of this matrix is represented as a cubic equation:

$$|J(V_3 \circ V_4) - \lambda I| = \begin{vmatrix} A-\lambda & B & C \\ D & K-\lambda & L \\ M & N & O-\lambda \end{vmatrix} = 0.$$

$$-\lambda^3 + (A+K+O)\lambda^2 - (AK+OA-MC-LN-DB)\lambda + AKO + DNC + BLM - MCK - ALN - OBD = 0,$$

where $J(V_3 \circ V_4)$ is the Jacobi matrix of operator (2.6), I is the unit matrix, and λ is the eigenvalue of the Jacobi matrix.

For a fixed point A_0, the solution of this cubic equation looks like this:

$$\lambda_1 = 1, \lambda_2 = \frac{\left(a\sqrt{5} - a + \sqrt{2(3-\sqrt{5})(a+1)^2} - 3\sqrt{5} + 11\right)}{2}, \lambda_3 = \frac{\left(a\sqrt{5} - a - \sqrt{2(3-\sqrt{5})(a+1)^2} - 3\sqrt{5} + 11\right)}{2}.$$

Now we introduce the Jacobi matrix for operator (2.7):

$$J(V_4 \circ V_3) = \begin{pmatrix} A' & B' & C' \\ D' & K' & L' \\ M' & N' & O' \end{pmatrix},$$

where

$$A' = x(-ay-z)(1+ay+z) + (1+ay+z)(1+y(1-ax-z)+z(1-x+y)),$$

$$B' = x(1-ax)(1+ay+z) + ax(1+y(1-ax-z)+z(1-x+y)),$$

$$C' = x(1-x)(1+ay+z) + x(1+y(1-ax-z)+z(1-x+y)),$$

$$D' = y(1-ax-z)(1-ay-2z) - ay(1-x(1+ay+z)+z(1-x+y)),$$

$$K' = y(1-ax-z)(z-ax) + (1-ax-z)(1-x(1+ay+z)+z(1-x+y)),$$

$$L' = y(1-2x+y)(1-ax-z) - y(1-x(1+ay+z)+z(1-x+y)),$$

$$M' = z(-1-z)(1-x+y) - z(1-y(1-ax-z)-x(1+ay+z)),$$

$$N' = z(z-1)(1-x+y) + z(1-y(1-ax-z)-x(1+ay+z)),$$

$$O' = z(y-x)(1-x+y) + (1-x+y)(1-y(1-ax-z) - x(1+ay+z)).$$

The Jacobian of operator (2.7) is also represented by the cubic equation:

$$|J(V_4 \circ V_3) - \lambda I| = \begin{vmatrix} A'-\lambda & B' & C' \\ D' & K'-\lambda & L' \\ M' & N' & O'-\lambda \end{vmatrix} = 0.$$

$$-\lambda^3+(A'+K'+O')\lambda^2-(A'K'+O'A'-M'C'-L'N'-D'B')\lambda+A'K'O'+D'N'C'+B'L'M'-M'C'K'-A'L'N'-O'B'D'=0.$$

Here also $J(V_4 \circ V_3)$ is the Jacobi matrix of operator (2.7), I is the unit matrix, and λ is the eigenvalue of the Jacobi matrix. Eigenvalue, i.e. the solution of this equation at a fixed point A_1 :

$$\lambda_1 = 1, \lambda_2 = 6 - 5\sqrt{5}, \lambda_3 = (1-\sqrt{5})(-\sqrt{5}-a).$$

It is not difficult to see that for any value of the parameter a, the fixed point A_1 is repulsive. As a result, we constructively proved the statement:

The fixed point A_0 of operator (2.6), and the fixed point A_1 of operator (2.7), are repulsive fixed points, for any parameter value $0 < a \leq 1$.

Now let us move on to the mappings of Lotka–Volterra of the following types:

$$V_5 : \begin{cases} x' = x(1+y+az), \\ y' = y(1-x-z), \\ z' = z(1-ax+y). \end{cases} \quad V_4 : \begin{cases} x' = x(1+y+z), \\ y' = y(1-x+z), \\ z' = z(1-x-y). \end{cases} \tag{2.9}$$

The compositions of these operators have the following form:

$$V_5 \circ V_4 : \begin{cases} x' = x(1+y+z)(1+y(1-x+z)+az(1-x-y)), \\ y' = y(1-x+z)(1-x(1+y+z)-z(1-x-y)), \\ z' = z(1-x-y)(1-ax(1+y+z)+y(1-x+z)). \end{cases} \tag{2.10}$$

$$V_4 \circ V_5 : \begin{cases} x' = x(1+y+az)(1+y(1-x-z)+z(1-ax+y)), \\ y' = y(1-x-z)(1-x(1+y+az)+z(1-ax+y)), \\ z' = z(1-ax+y)(1-x(1+y+az)-y(1-x-z)). \end{cases} \tag{2.11}$$

Here $0 \leq a \leq 1$.

For both compositions, there is a fixed point on the edge of Γ_{23}, for operator (2.10) the fixed point is a point $B_0\left(0; \frac{3-\sqrt{5}}{2}; \frac{\sqrt{5}-1}{2}\right)$, for operator (2.11), a fixed point is a point $B_1\left(0; \frac{\sqrt{5}-1}{2}; \frac{3-\sqrt{5}}{2}\right)$.

Now, to find out the characters of these fixed points, for each of these compositions, we will make a Jacobi matrix. For operator (2.10), the Jacobi matrix has the following form:

$$J(V_5 \circ V_4) = \begin{pmatrix} A & B & C \\ D & K & L \\ M & N & O \end{pmatrix},$$

where

$$A = x(1+y+z)(-az-y)+(1+y+z)(1+az(1-x-y)+y(1-x+z)),$$

$$B = x(1+y+z)(1-x-az)+x(1+az(1-x-y)+y(1-x+z)),$$

$$C = x(1+y+z)(a(1-x-y)+y)+x(1+az(1-x-y)+y(1-x+z)),$$

$$D = y(-y-1)(1-x+z)-y(1-z(1-x-y)-x(1+y+z)),$$

$$K = y(z-x)(1-x+z)+(1-x+z)(1-z(1-x-y)-x(1+y+z)),$$

$$L = y(y-1)(1-x+z)+y(1+z(x+y-1)-x(1+y+z)),$$

$$M = z(1-x-y)(-a(1+y+z)-y)-z(1-ax(1+y+z)+y(1-x+z)),$$

$$N = z(1-x-y)(1-ax-x+z)-z(-ax(1+y+z)+y(1-x+z)+1),$$

$$O = z(1-x-y)(y-ax)+(1-x-y)(1-ax(1+y+z)+y(1-x+z)).$$

The Jacobian of this matrix looks like this:

$$|J(V_5 \circ V_4)-\lambda I| = \begin{vmatrix} A-\lambda & B & C \\ D & K-\lambda & L \\ M & N & O-\lambda \end{vmatrix} = 0.$$

$$-\lambda^3+(A+K+O)\lambda^2-(AK+OA-MC-LN-DB)\lambda+AKO+DNC+BLM-MCK-ALN-OBD=0.$$

Here $J(V_5 \circ V_4)$ is the Jacobi matrix of the operator (2.10), I is the unit matrix, λ is the eigenvalue of the Jacobi matrix. Eigenvalues, i.e. the solution of this equation at the point B_0

$$\lambda_1 = 1, \quad \lambda_2 = 6-2\sqrt{5}, \quad \lambda_3 = (3-\sqrt{5})a+\sqrt{5}+1.$$

For (2.11), the Jacobi matrix

$$J(V_4 \circ V_5) = \begin{pmatrix} A' & B' & C' \\ D' & K' & L' \\ M' & N' & O' \end{pmatrix},$$

where

$$A' = x(-az-y)(1+y+az)+(1+y+az)(1+y(1-x-z)+z(1-ax+y)),$$

$$B' = x(1-x)(1+y+az)+x(1+y(1-x-z)+z(1-ax+y)),$$

$$C' = x(1-ax)(1+y+az)+ax(1+y(1-x-z)+z(1-ax+y)),$$

$$D' = y(1-x-z)(-1-y-2az)-y(1-x(1+az+y)+z(1-ax+y)),$$

$$K' = y(1-x-z)(z-x)+(1-x-z)(1-x(1+y+az)+z(1-ax+y)),$$

$$L' = y(1-x-z)(1-2ax+y)-y(1-x(1+y+az)+z(1-ax+y)),$$

$$M' = z(-1-az)(1-ax+y)-az(1-y(1-x-z)-x(1+y+az)),$$

$$N' = z(z-1)(1-ax+y) + z(1-y(1-x-z) - x(1+y+az)),$$

$$O' = z(y-ax)(1-ax+y) + (1-ax+y)(1-y(1-x-z) - x(1+y+az)).$$

Here the Jacobian will also be a cubic equation.

$$|J(V_4 \circ V_5) - \lambda I| = \begin{vmatrix} A'-\lambda & B' & C' \\ D' & K'-\lambda & L' \\ M' & N' & O'-\lambda \end{vmatrix} = 0.$$

$$-\lambda^3 + (A'+K'+O')\lambda^2 - (A'K'+O'A'-M'C'-L'N'-D'B')\lambda + A'K'O'+D'N'C'+B'L'M'-M'C'K'-A'L'N'-O'B'D' = 0.$$

Here, as in the previous cases, $J(V_4 \circ V_5)$ is the Jacobi matrix for (2.11), I is the unit matrix, and λ is the eigenvalue of the Jacobi matrix.

Find the eigenvalues for a fixed point B_1 :

$$\lambda_1 = 1, \quad \lambda_2 = 6 - 2\sqrt{5}, \quad \lambda_3 = (3-\sqrt{5})a + \sqrt{5} + 1.$$

As a result, similar to the previous case, we obtained confirmation:

The fixed point B_0 of operator (2.10), and the fixed point B_1 of operator (2.11), are repulsive fixed points for any parameter value $0 < a \leq 1$.

As a result, we proved that the card of fixed points of the Lotka–Volterra mappings $V_3 \circ V_4, V_4 \circ V_3, V_5 \circ V_4$, and $V_4 \circ V_5$ has the form as in Figure 1 case ***i***).

Now let us move on to the following operators. Here we introduce a coefficient connecting mutually inversely directed edges, i.e. $0 < a_{23} = a \leq 1$.

$$V_6 : \begin{cases} x' = x(1+y+z), \\ y' = y(1-x-az), \\ z' = z(1-x+ay). \end{cases} \quad V_4 : \begin{cases} x' = x(1+y+z), \\ y' = y(1-x+z), \\ z' = z(1-x-y). \end{cases} \tag{2.12}$$

The composition of these mappings has the following form:

$$V_6 \circ V_4 : \begin{cases} x' = x(1+y+z)(1+y(1-x+z)+z(1-x-y)), \\ y' = y(1-x+z)(1-x(1+y+z)-az(1-x-y)), \\ z' = z(1-x-y)(1-x(1+y+z)+ay(1-x+z)). \end{cases} \tag{2.13}$$

$$V_4 \circ V_6 : \begin{cases} x' = x(1+y+z)(1+y(1-x-az)+z(1-x+ay)), \\ y' = y(1-x-az)(1-x(1+y+z)+z(1-x+ay)), \\ z' = z(1-x+ay)(1-x(1+y+z)-y(1-x-az)). \end{cases} \tag{2.14}$$

Fixed points of compositions $V_6 \circ V_4$ and $V_4 \circ V_6$, belonging to the edge Γ_{23}, are found by solving systems by taking $x = 0$:

$$\begin{cases} 1 = (1+z)(1-az(1-y)), \\ 1 = (1-y)(1+ay(1+z)), \\ 1 = y+z. \end{cases} \quad \begin{cases} 1 = (1-az)(1+z(1+ay)), \\ 1 = (1+ay)(1-y(1-az)), \\ 1 = y+z. \end{cases}$$

A fixed point belonging to the edge Γ_{23} of the mapping (2.13) – $C_0\left(0; \frac{3\sqrt{a}-\sqrt{a+4}}{2\sqrt{a}}; \frac{\sqrt{a+4}-\sqrt{a}}{2\sqrt{a}}\right)$, a fixed point for the mapping (2.14) – $C_1\left(0; \frac{a+\sqrt{a(a+4)}-2}{2a}; \frac{a-\sqrt{a(a+4)}+2}{2a}\right)$.

The Jacobi matrix for the mapping (2.13):

$$J(V_6\circ V_4)=\begin{pmatrix} A & B & C\\ D & K & L\\ M & N & O\end{pmatrix},$$

where

$$A=x(1+y+z)(-z-y)+(1+y+z)(1+z(1-x-y)+y(1-x+z)),$$

$$B=x(1-x)(1+y+z)+x(1+z(1-x-y)+y(1-x+z)),$$

$$C=x(1-x)(1+y+z)+x(1+z(1-x-y)+y(1-x+z)),$$

$$D=y(1-x+z)(az-y-z-1)-y(1-az(1-x-y)-x(1+y+z)),$$

$$K=y(az-x)(1-x+z)+(1-x+z)(1-az(1-x-y)-x(1+y+z)),$$

$$L=y(-x+z+1)(-a(1-x-y)-x)+y(1-az(1-x-y)-x(1+y+z)),$$

$$M=z(1-x-y)(-ay-y-z-1)-z(1-x(1+y+z)+ay(1-x+z)),$$

$$N=z(1-x-y)(a(1-x+z)-x)-z(1-x(1+y+z)+ay(1-x+z)),$$

$$O=z(1-x-y)(ay-x)+(1-x-y)(1-x(1+y+z)+ay(1-x+z)).$$

The Jacobian of this matrix is also a cubic equation

$$|J(V_6\circ V_4)-\lambda I|=\begin{vmatrix} A-\lambda & B & C\\ D & K-\lambda & L\\ M & N & O-\lambda\end{vmatrix}=0.$$

$$-\lambda^3+(A+K+O)\lambda^2-(AK+OA-MC-LN-DB)\lambda+AKO+DNC+BLM-MCK-ALN-OBD=0.$$

Here also $J(V_6\circ V_4)$ -is the Jacobi matrix of the operator (2.13), I is the unit matrix, and λ is the eigenvalue of the matrix.

Eigenvalues at a fixed point C_0

$$\lambda_1=1,\quad \lambda_2=4,\quad \lambda_3=\frac{2a^2-a\sqrt{\frac{a}{a+4}}-a\sqrt{\frac{(a+4)(2a+1)^2}{a}}+10a-4\sqrt{\frac{a}{a+4}}}{2a}.$$

For (2.14) operator, the Jacobi matrix has the form:

$$J(V_4\circ V_6)=\begin{pmatrix} A' & B' & C'\\ D' & K' & L'\\ M' & N' & O'\end{pmatrix},$$

where

$$A' = x(-z-y)(1+y+z) + (1+y+z)(1+y(1-x-az)+z(1-x+ay)),$$

$$B' = x(1-x)(1+y+z) + x(1+y(1-x-az)+z(1-x+ay)),$$

$$C' = x(1-x)(1+y+z) + x(1+y(1-x-az)+z(1-x+ay)),$$

$$D' = y(1-x-az)(-1-y-2z) - y(1-x(1+z+y)+z(1-x+ay)),$$

$$K' = y(1-x-az)(az-x) + (1-x-az)(1-x(1+y+z)+z(1-x+ay)),$$

$$L' = y(1-x-az)(1-2x+ay) - ay(1-x(1+y+z)+z(1-x+ay)),$$

$$M' = z(-1-z)(1-x+ay) - z(1-y(1-x-az)-x(1+y+z)),$$

$$N' = z(az-1)(1-x+ay) + az(1-y(1-x-az)-x(1+y+z)),$$

$$O' = z(ay-x)(1-x+ay) + (1-x+ay)(1-y(1-x-az)-x(1+y+z)).$$

The Jacobian of this matrix looks like this:

$$|J(V_4 \circ V_6) - \lambda I| = \begin{vmatrix} A'-\lambda & B' & C' \\ D' & K'-\lambda & L' \\ M' & N' & O'-\lambda \end{vmatrix} = 0.$$

$$-\lambda^3 + (A'+K'+O')\lambda^2 - (A'K'+O'A'-M'C'-L'N'-D'B')\lambda + A'K'O' + D'N'C' + B'L'M' - M'C'K' - A'L'N' - O'B'D' = 0.$$

Here $J(V_4 \circ V_6)$ is the Jacobi matrix for (2.14), I is the unit matrix, and λ is the eigenvalue of the matrix.

The eigenvalue at the point C_1 :

$$\lambda_1 = 4,$$

$$\lambda_2 = \frac{a^2 + \sqrt{2a^4 - 2\sqrt{a(a+4)}a^3 - 2a^3 + 6\sqrt{a(a+4)}a^2 - 23a^2 + 8a\sqrt{a(a+4)} - 4a} - a\sqrt{a(a+4)} + 6a - \sqrt{a(a+4)}}{2a},$$

$$\lambda_3 = \frac{a^2 - \sqrt{2a^4 - 2\sqrt{a(a+4)}a^3 - 2a^3 + 6\sqrt{a(a+4)}a^2 - 23a^2 + 8a\sqrt{a(a+4)} - 4a} - a\sqrt{a(a+4)} + 6a - \sqrt{a(a+4)}}{2a}.$$

Theorem 2.5 *Let the mappings be given V_4 and V_6.*

1. Composite operators (2.13) and (2.14) have four fixed points each:

– these are the vertices of the simplex e_1, e_2, e_3;

– fixed point belonging to the edge Γ_{23} to mapping (2.13) – $C_0\left(0; \frac{3\sqrt{a}-\sqrt{a+4}}{2\sqrt{a}}; \frac{\sqrt{a+4}-\sqrt{a}}{2\sqrt{a}}\right)$, and for mapping (2.14) – $C_1\left(0; \frac{a+\sqrt{a(a+4)}-2}{2a}; \frac{a-\sqrt{a(a+4)}+2}{2a}\right)$.

2. Fixed point C_0 by $0 < a < \frac{1}{12}(5\sqrt{33}-27)$ and $\frac{1}{2} < a < 1$ is repulsive, and when $\frac{1}{12}(5\sqrt{33}-27) < a < \frac{1}{2}$ is saddled.

3. Fixed point C_1 by $a = \frac{1}{2}$ and $a \approx 0,14$ is repulsive, and when $a \in (0;0,14) \cup (0,14;0,5) \cup (0,5;1)$ is saddled.

*Card of fixed points of the composition $V_6 \circ V_4$ at coefficient values $0 < a < \frac{1}{12}(5\sqrt{33}-27)$ and $\frac{1}{2} < a < 1$ has the form of a case **i)** from Figure 1, and in other cases has the form **ii)**.*

*Card of fixed points of the composition $V_4 \circ V_6$ at coefficient values $a = \frac{1}{2}$ and $a \approx 0,14$ it has the form from Figure 1 of the case **i)**, and when $a \in (0;0,14) \cup (0,14;0,5) \cup (0,5;1)$ case **iii)**.*

Now we introduce the coefficients $a_{ki} = a$ and show that this case is similar to the previous one:

$$V_7 : \begin{cases} x' = x(1 + ay + az), \\ y' = y(1 - ax - az), \\ z' = z(1 - ax + ay). \end{cases} \quad V_4 : \begin{cases} x' = x(1 + y + z), \\ y' = y(1 - x + z), \\ z' = z(1 - x - y). \end{cases} \tag{2.15}$$

The composition of these mappings is represented as:

$$V_7 \circ V_4 : \begin{cases} x' = x(1 + y + z)(1 + ay(1 - x + z) + az(1 - x - y)), \\ y' = y(1 - x + z)(1 - ax(1 + y + z) - az(1 - x - y)), \\ z' = z(1 - x - y)(1 - ax(1 + y + z) + ay(1 - x + z)). \end{cases} \tag{2.16}$$

$$V_4 \circ V_7 : \begin{cases} x' = x(1 + ay + az)(1 + y(1 - ax - az) + z(1 - ax + ay)), \\ y' = y(1 - ax - az)(1 - x(1 + ay + az) + z(1 - ax + ay)), \\ z' = z(1 - ax + ay)(1 - x(1 + ay + az) - y(1 - ax - az)). \end{cases} \tag{2.17}$$

Compositions $V_7 \circ V_4$ and $V_4 \circ V_7$ have four fixed points:

– these are the vertices of the simplex e_1, e_2, e_3,

– point $A_0\left(0; \frac{3\sqrt{a}-\sqrt{a+4}}{2\sqrt{a}}; \frac{\sqrt{a+4}-\sqrt{a}}{2\sqrt{a}}\right)$ and a fixed point $A_1\left(0; \frac{a+\sqrt{a(a+4)}-2}{2a}; \frac{a-\sqrt{a(a+4)}+2}{2a}\right)$ for each of the operators, respectively.

If $0,5 \le a \le 1$, then the fixed point $A_0 \in S^2$, i.e. this point belongs to the edge $A_0 \in \Gamma_{23}$, this is the case ***i)***. If $0 < a < 0,5$, then $A_0 \notin S^2$ and the card looks like cases ***ii)*** or ***iii)***.

Theorem 2.6 *For compositions $V_7 \circ V_4$ and $V_4 \circ V_7$: if $0 < a < \frac{1}{12}(5\sqrt{33}-27), \frac{1}{2} < a < 1$, then the fixed point A_0, accordingly, the fixed point A_1, are repulsive if $\frac{1}{12}(5\sqrt{33}-27) < a < \frac{1}{2}$, then they are saddled.*

Proof The theorem can be proved, as in the previous cases, by analyzing the spectrum of the Jacobian, in accordance with Definitions 2.2, 2.3, and 2.4. For the composition $V_7 \circ V_4$, the eigenvalues of the Jacobi matrix look like:

$$\lambda_1 = 1, \quad \lambda_2 = 2(a+1), \quad \lambda_3 = \frac{2a^2 - \sqrt{\frac{a}{a+4}}a - \sqrt{\frac{(a+4)(2a+1)^2}{a}}a + 10a - 4\sqrt{\frac{a}{a+4}}}{2a}.$$

For the composition $V_4 \circ V_7$, the eigenvalues are:

$$\lambda_1 = 1, \quad \lambda_2 = 2(a+1), \quad \lambda_3 = -a\sqrt{\frac{a+4}{a}} + a - \sqrt{\frac{a+4}{a}} + 5.$$

Now let us move on to a more general case:

$$V_8 : \begin{cases} x' = x(1+ay+az), \\ y' = y(1-ax-az), \\ z' = z(1-ax+ay). \end{cases} \quad V_9 : \begin{cases} x' = x(1+by+bz), \\ y' = y(1-bx+bz), \\ z' = z(1-bx-by). \end{cases} \tag{2.18}$$

here $0 < a, b \leq 1$.

Compositions of these mappings

$$V_8 \circ V_9 : \begin{cases} x' = x(1+by+bz)(1+ay(1-bx+bz)+az(1-bx-by)), \\ y' = y(1-bx+bz)(1-ax(1+by+bz)-az(1-bx-by)), \\ z' = z(1-bx-by)(1-ax(1+by+bz)+ay(1-bx+bz)). \end{cases} \tag{2.19}$$

$$V_9 \circ V_8 : \begin{cases} x' = x(1+ay+az)(1+by(1-ax-az)+bz(1-ax+ay)), \\ y' = y(1-ax-az)(1-bx(1+ay+az)+bz(1-ax+ay)), \\ z' = z(1-ax+ay)(1-bx(1+ay+az)-by(1-ax-az)). \end{cases} \tag{2.20}$$

Fixed points of operators $V_8 \circ V_9$ and $V_9 \circ V_8$, accordingly

$$A_0 \left(0; \frac{b\sqrt{a} - \sqrt{b(ab+4)} + 2\sqrt{a}}{2b\sqrt{a}}; \frac{b\sqrt{a} + \sqrt{b(ab+4)} - 2\sqrt{a}}{2b\sqrt{a}}\right)$$

and

$$A_1 \left(0; \frac{a\sqrt{b} - \sqrt{a(ab+4)} - 2\sqrt{b}}{2a\sqrt{b}}; \frac{a\sqrt{b} + \sqrt{a(ab+4)} + 2\sqrt{b}}{2a\sqrt{b}}\right).$$

Theorem 2.7 *The fixed point A_0 of the operator $V_8 \circ V_9$, and also the fixed point A_1 of the operator $V_9 \circ V_8$ at $b > \frac{a}{a+1}$ are located on the edge of the simplex, and in other cases fixed points are located outside the simplex and these points are repulsive at these coefficient values.*

Proof To prove the theorem, we check the Jacobian spectra of skew-symmetric mapping matrices. The eigenvalues of the mapping are equal to: $V_8 \circ V_9$

$$\lambda_1 = 1,$$

$$\lambda_2 = (bx - by - bz - 1)(2abxy + 2abxz + ax - ay - az - 1),$$

$$\lambda_3 = (bx + by - bz - 1)(2abxy - 2abxz + ax - ay + az - 1).$$

The eigenvalues of the mapping $V_9 \circ V_8$ are equal to:

$$\lambda_1 = 1,$$

$$\lambda_2 = (ax - ay - az - 1)(2abxy + 2abxz + bx - by - bz - 1),$$
$$\lambda_3 = (ax - ay + az - 1)(2abxz - 2abyz + bx + by - bz - 1).$$

Substituting the coordinates of the fixed points, it is not difficult to observe that in both cases $|\lambda_2| > 1$ and $|\lambda_3| > 1$. This indicates that both fixed points are repulsive.

As a result, we proved that the fixed point cards of compositions $V_8 \circ V_9$ and $V_9 \circ V_8$ by $b > \frac{a}{a+1}$ and $0 < a \leq 1$ look like the case in Figure 1 ***i)***, while in other cases they look like ***ii)***.

3. Conclusion and discussion

In the paper, we study the full dynamics of the composition of the Lotka–Volterra mappings corresponding to transitive tournaments. Fixed points are found, cards of fixed points are constructed, and also criteria and characters of these fixed points are given. The study of the dynamics of the internal points of the composition of discrete Lotka–Volterra mappings is relevant, as they can be viewed as a discrete model for studying epidemiological situations, particularly the dynamics of the spread of sexually transmitted viral diseases among the population. Each operator in the composition of these mappings represents the total population, comprising both male and female populations. In [11, 13], continuous models for studying epidemiological situations are considered. In [14], it is proposed to apply multicriteria group decision making (MCGDM) to COVID-19 using bipolar soft ideal coarse sets with the help of two methods, but we believe that the discrete model describes the picture more adequately than the continuous one. The epidemiological significance of the composition will be described in detail in subsequent works.

References

[1] Andronov AA, Leontovich EA. Gordon II and Mayer AG. Qualitative Theory of Dynamical Systems of the Second Order. Moscow, Nauka Publishing House, 1966.

[2] Eshmamatova DB, Ganikhodzhaev RN. Tournaments of Volterra type transversal operators acting in the simplex S^{m-1}. International AIP Conference Proceedings; Tashkent, Uzbekistan; 2021; 2365 (1): 060009-1-06000-6. https://doi.org/10.1063/5.0057303

[3] Eshmamatova DB, Tadzhieva MA, Ganikhodzhaev RN. Criteria for internal fixed points existence of discrete dynamic Lotka–Volterra systems with homogeneous tournaments. Izvestiya Vysshikh Uchebnykh Zavedeniy. Prikladnaya Nelineynaya Dinamika 2022; 30 (6): 702-716.

[4] Ganikhodzhaev RN. Quadratic stochastic operators, Lyapunov function and tournaments. Sbornik Mathematics 1993; 76 (2): 489-506.

[5] Ganikhodzhaev RN, Tadzhieva MA and Eshmamatova DB. Dynamical proporties of quadratic homeomorphisms of a finite-dimensional simplex. Journal of Mathematical Sciences 2020; 245 (3): 398-402.

[6] Ganikhodzhaev RN, Eshmamatova DB. Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories. Vladikavkazskii Matematicheskii Zhurnal 2006; 8 (2): 12-28.

[7] Gantmacher FR. The Theory of Matrices. New York: Chelsea Publishing, 1959.

[8] Ganikhodzhaev RN. A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems. Matematicheskie Zametki 1994; 56 (5-6): 1125-1131.

[9] Harary F. Graph Theory. Addison-Wesley Publishing Company, 1969.

[10] Harary F and Palmer EM. Graphical enumeration. Academic Press, New York and London, 1972.

[11] Kermack W and McKendrick AA. A contribution to the mathematical theory of epidemics. Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 1927; 115 (772): 700-721.

[12] Moon JW. Topics on Tournaments. New York: Holt Renehart and Winston, 2013.

[13] Murray JD. Mathematical Biology. New York, Springer-Verlag, 3d edition, 2007.

[14] Mustafa, Heba I. Bipolar soft ideal rough set with applications in COVID-19. Turkish Journal of Mathematics 2023; 47 (1): 1-36. https://doi.org/10.55730/1300-0098.3343